CONVERSES OF GAUSS' THEOREM ON THE ARITHMETIC MEAN*

BY

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1.1. **Introduction.** Let T denote a domain (open continuum) in the plane. Let u be a function, continuous in T, and equal, at each point P of T, to the arithmetic mean of its values on each circle with center at P and lying, with its interior, in T. According to Gauss' theorem, this is the property of any function u harmonic in T. On the other hand, Koebe† showed, that conversely, any function with the stated properties is harmonic in T. It is with some extensions of this theorem of Koebe we are concerned.

Undoubtedly, one of the simplest proofs of Koebe's theorem is as follows. Let c be any circle lying with its interior C in T. Let m be the minimum of u in $C+c$. Then, since u is continuous in $C+c$, either u assumes the value m on c, or else there is a point P of C such that $u=m$ at P, and $u>m$ at all points of $C+c$ nearer to c than is P. But, by the mean-value property, the second of these two alternatives is impossible. Thus, u assumes its lower bound in $C+c$ on c. Applying this result to $u-v$ and to $v-u$, where v is the function harmonic in C, continuous on $C+c$, and equal to u on C, we conclude that u is harmonic in C, and therefore in T.

This reasoning can evidently be applied in case the circles attached to each point P, on which the arithmetic mean of u is equal to the value of u at their center P, constitute any infinite set whose radii have 0 as lower limit. It fails, of course, if these radii are bounded away from 0. The question then arises as to what can be said about a function which coincides at each point P with its arithmetic mean on some single circle about P as center. In what follows we shall be concerned with functions satisfying a condition of this type.

1.2. All our theorems are valid both in the plane and in space. The proofs themselves are independent of the number of dimensions if "circle" is interpreted as "sphere" and "plane" as "space" in the case of three dimensions. We shall always suppose that T is a bounded domain. Results of a similar character are readily obtained by inversions for unbounded domains, containing or not the point at infinity, provided these domains do not, with their

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† Koebe, **6**. Numbers in heavy type refer to the bibliography at the end of this paper.

boundaries, fill out the extended plane. The results of some of the theorems can easily be extended to domains not having this latter property. This does not, however, seem to be the case for those of others. In Theorem V, for example, a condition equivalent to boundedness seems to be essential.

We denote by t the boundary of T. We associate with each point P of T a normal* domain $C(P)$, containing P, and consisting wholly of points of T. We denote by $c(P)$ the boundary of $C(P)$. This boundary may coincide wholly or in part with t. We denote by τ the set of those points common to t and some $c(P)$.

We denote, for a function u continuous in $T+\tau$, by $A\{u(P)\}$, or by $A(u)$, the value at P of the function harmonic in $C(P)$, continuous in $C(P)+c(P)$, and equal to u on $c(P)$. Our theorems are concerned chiefly with a continuous function u satisfying $u(P)=A(u)$ in T. This condition is, of course, a generalization of the condition that u be equal at each point P to the arithmetic mean of its values on some circle about P as center. In fact, when $c(P)$ is a circle about P as center, then $A\{u(P)\}$ is the arithmetic mean of u on $c(P)$.

2.1. **On the bounds of a function satisfying $u=A(u)$.** We first apply the reasoning instrumental in the preceding proof of Koebe's theorem. We obtain

THEOREM I. *Let u be continuous in $T+\tau$. Then, if $u(P)\geqq A(u)$ in T, it follows that u tends to its lower bound in T on a sequence of points in T tending to a point of t. Similarly, if $u\leqq A(u)$, then u tends to its upper bound in T on a sequence of points in T tending to a point of t.*

We need consider only the first part of this theorem. Let m denote the lower bound of u in T; and let λ be the set of points in $T+\tau$ on which $u=m$. If λ is void, or if λ contains any point of t, the conclusion follows from the continuity of u in $T+\tau$. Let us suppose then that λ is not void and that it lies wholly in T. Let P be a point of λ. Then, by a familiar property of harmonic functions, and our hypothesis on u, it follows that all the points of $c(P)$ are points of λ. Thus, $u=m$ on a point nearer to t than is P. We conclude that the distance from λ to t is 0; for if it were positive the set λ would be closed and we could select, contrary to fact, a point P of λ such that no other point of λ lies nearer to t than does P. This proves the theorem.

2.2. A different form of reasoning gives a somewhat stronger result than that embodied in Theorem I. It can be shown in fact that, *under the first hypotheses of that theorem, if u attains in T its lower bound, u attains that*

* A domain C is normal if the Dirichlet problem for C admits a solution for every assigned function continuous on the boundary of C.

For a set of references on the Dirichlet problem and on harmonic functions in general, the reader is referred to Kellogg, **2** and **4**.

bound in $T+\tau$ in every neighborhood of every point of t. This is a consequence of the following theorem. We note also, as another corollary of this theorem, that *if u is continuous in $T+t$ and satisfies $u(P)=A(u)$ in T, then u cannot assume in T both its bounds without reducing to a constant.*

THEOREM II. *Let u be continuous in $T+\tau$. Then, if u attains in T its lower bound m in T, and if $u(P)\geqq A(u)$ in T, it follows that every point of T is a point of some $C(P)$ on the boundary of which $u=m$ identically.*

Let P_1 be a point of T at which $u=m$. Then plainly $u\equiv m$ on $c(P_1)$. Now let P_2 be any second point of T. If $u(P_2)=m$, then $u\equiv m$ on $c(P_2)$ and there is nothing to prove. In the contrary case, let α be a polygonal line, lying in T, and having its end points at P_1 and P_2. Consider the points of α at which $u=m$. This set is a non-null closed set. We therefore can select the last point P^*, in the sense P_1 to P_2, at which $u=m$. On $c(P^*)$ we have $u\equiv m$. Accordingly, $c(P^*)$ cannot cut α between P^* and P_2. Thus, P_2 is a point of $C(P^*)$. We conclude the truth of the theorem.

2.3. The question now arises whether, in contradistinction to non-constant harmonic functions, a non-constant function having the generalized mean-value property can attain one or both of its bounds without reducing to a constant. The answer is that in general such a function can assume both its bounds. Consider in fact the following example.

Let O be any fixed point. Let T be the interior of the unit circle about O as center. Let b_n, $n=2, 3, \cdots$, be the circle of radius $1-1/n$ about O. Let $u(P)=0$ for $OP<1/2$ and for P on $b_2, b_4, \cdots$; and let $u(P)=1$ for P on $b_3, b_5, \cdots$. Finally, let u be harmonic in the region bounded by b_n, b_{n+1} and assume continuously the values defined for u on b_n and b_{n+1}. Then, noting that, if P is on b_n, $n=2, 3, \cdots$, we can take for $C(P)$ the interior of b_{n+2}, and that if P is not on one of the circles b_n we can take for $C(P)$ the interior of any sufficiently small circle about P, we see that u is continuous in T and has the generalized mean-value property. In addition, we see that u assumes in T its lower bound 0 and its upper bound 1.

2.4. Considerably more information can be obtained in regard to the question raised in the preceding paragraph. It can be shown that a function u of the prescribed type cannot attain both its bounds if the domains $C(P)$ are of a sufficiently restricted character. We consider in the next theorem a set of domains satisfying the following conditions:

(a) *the diameter $\delta(P)$ of $C(P)$ tends to 0 as P tends to any point of t;*

(b) *the boundary $c(P)$ has at least one point in common with $c(Q)$ if there is a point of $c(P)$ exterior and a point of $c(P)$ interior to $C(Q)$.*

The second of these two conditions is satisfied, of course, if each $c(P)$ is a connected set. Both conditions are satisfied if each $c(P)$ is a circle about P as center.

THEOREM III. *Let the domains* $C(P)$ *satisfy the conditions* (a) *and* (b). *Let* u *be continuous in* $T+\tau$ *and let* $u(P)=A(u)$ *in* T. *Then, if the bounds of* u *in* T *are distinct, both these bounds cannot be attained by* u *in* T.

To prove this we show that the contrary assumption, that u attains in T both its bounds, $m<M$, implies a contradiction. We first select a point P_1 at which $u=M$, and then choose a point Q of t such that the segment P_1Q lies in T except for its extremity Q. We note that Q is not a point of $c(P)$ for any P in T; for if it were we should have, contrary to hypothesis, $m=M$, because u would be continuous at Q and would tend, according to our remark in §2.2, to M on one sequence, and to m on another sequence, of points tending to Q.

Consider now $c(P_1)$. On this set we have $u\equiv M$. Further, $c(P_1)$ has at least one point in common with P_1Q. We denote by P_2 one such point, observing that P_2 is a point of T.

Consider next the set of points on P_2Q at which $u=m$. Since P_2 is a point of some $C(P)$ on the boundary of which $u\equiv m$, this set is not void. In addition, if we adjoin to this set the point Q, the resulting set is closed. Accordingly, there is a first point, starting from P_2, of P_2Q at which $u=m$. We denote this point by P_3, observing that P_3 is a point of T. Now at P_2, $u=M$, and on $c(P_3)$, $u\equiv m<M$. It follows from this, and our choice of P_3, that P_2 is a point of $C(P_3)$. We deduce further, on applying the fact that $u\equiv M$ on $c(P_1)$ and the fact that the domains $C(P)$ satisfy condition (b), that $c(P_1)$ is contained in $C(P_3)$. Thus,

$$\delta(P_1) \leqq \delta(P_3).$$

We continue this process. We select a point P_4 of intersection of P_3Q with $c(P_3)$, noting that P_4 is a point of T. We next select the first point, starting from P_4, of P_4Q at which $u=M$. We denote this point by P_5, observing that $C(P_5)$ contains P_3. We find also that $C(P_5)$ contains $c(P_3)$; and accordingly that

$$\delta(P_1) \leqq \delta(P_3) \leqq \delta(P_5).$$

Proceeding in this manner we obtain an infinite sequence of points $P_1, P_2, \cdots$, lying on P_1Q and in T. We have $P_1P_2<P_1P_3< \cdots <P_1Q$, and $u(P_1)=u(P_2)=u(P_5)= \cdots =M$, and $u(P_3)=u(P_4)=u(P_7)= \cdots =m$. We find also that

$$\delta(P_1) \leqq \delta(P_3) \leqq \cdots . \qquad (2.41)$$

We arrive now at the desired contradiction. It follows from (2.41), and the fact that the domains $C(P)$ satisfy the condition (a), that the points P_n do not tend to Q. Accordingly, these points have a limit point P in T. But this is impossible; for u is continuous at P, and is therefore distinct from at least one of its bounds in some neighborhood of P.

2.5. It is conceivable that, under the restrictions placed upon u in Theorem III, u can attain neither of its bounds in T without reducing to a constant. This however is not the case. An example illustrating this point follows.

We take for T the interior of the unit circle about some fixed point O as center. We shall so define u that it assumes its lower bound in T everywhere in T except in certain circles $k_1, k_2, \cdots$. The domain $C(P)$ will be for each P the interior of a circle about P as center.

Let Q be a point on the boundary of T. Let $P_1, P_2, \cdots$ be points on OQ such that

$$0 < OP_1 < OP_2 < \cdots < OQ, \qquad \lim_{n\to\infty} P_n = Q.$$

About P_n as center we now construct a circle d_n of radius ρ_n, choosing ρ_n so that each d_n lies in T, and is exterior to every other d_m of the set. The circle k_n, then, shall be the circle about P_n with radius

$$r_n = \rho_n(1 - OP_{n+1})/3.$$

The circles k_n have the following properties. Each is interior to T and each is exterior to every other circle of the set. Further, if P is interior to k_n, then the circle $e_1(P)$ with P as center and radius $2r_n$ lies exterior to k_n and interior to d_n, whereas the circle $e_2(P)$ with center at P and passing through P_{n+1} lies in T. We denote by K_n the interior of k_n.

Turning now to the definition of u, we first choose a set of positive numbers $B_1, B_2, \cdots$. We take $B_1=1$. We then choose B_2 so that the arithmetic mean on $e_2(P')$ of the function $u_2(P)$, defined as

$$B_2(r_2 - PP_2)$$

for P in K_2 and as 0 elsewhere, exceeds B_1 independently of the position of P' in K_1. This is possible since the mean in question exceeds a constant multiple of B_2 for all P' in K_1. Continuing in this manner we choose, in general, B_{n+1} so large that the arithmetic mean on $e_2(P')$ of the function $u_{n+1}(P)$, defined as

$$B_{n+1}(r_{n+1} - PP_{n+1})$$

for P in K_{n+1} and as 0 elsewhere, exceeds B_n independently of the position of P' in K_n.

We now define $u(P)$ as

$$B_n(r_n - PP_n)$$

for P in K_n, $n=1, 2, \cdots$, and as 0 at all other points of T. It is evident then that u is continuous in T and assumes in T its lower bound, 0. It is clear also that, if P is exterior to all the circles k_n, then u has the mean-value property at P with respect to all sufficiently small circles about P. It remains to consider the points of k_n+K_n.

Suppose first that P is on k_n. For such a point we can take for $C(P)$ the interior of the circle with P as center and with radius $2r_n$; for then $C(P)+c(P)$ lies in T and u is 0 at P and on $c(P)$.

Suppose now that P is in K_n. Consider the circles $e_1(P)$ and $e_2(P)$. On $e_1(P)$ the arithmetic mean $A_1(P)$ of u is 0; and on $e_2(P)$ the arithmetic mean $A_2(P)$ of u exceeds B_n because of our choice of the B_m. Thus,

$$A_1(P) < u(P) < A_2(P), \text{ since } 0<u(P)<B_n.$$

Now as η varies from η_1, the radius of e_1, to η_2, the radius of e_2, the circle of radius η about P as center remains in T, and the arithmetic mean of u on this circle varies continuously from A_1 to A_2. Hence we can select an η so that the arithmetic mean of u on the corresponding circle is exactly $u(P)$. The interior of this circle we can take for $C(P)$. The function u has then the mean-value property at P. Accordingly, u has all the asserted properties.

3.1. **Sufficient conditions that u be harmonic in T.** We turn now to a study of the conditions under which a function possessing the generalized mean-value property is necessarily harmonic. One result in this direction is readily obtained as a corollary of Theorem I.

THEOREM IV. *Let u be continuous in $T+\tau$, and satisfy $u(P)=A(u)$ in T. Then, if there exists a function v, harmonic in T, such that*

$$\lim_{P\to Q} \{u(P) - v(P)\} = 0 \qquad (P \text{ in } T)$$

at every point Q of t, it follows that u is harmonic in T.

We note, in fact, that $u-v$ is continuous in $T+t$ if defined as 0 on t. Accordingly, v is continuous on $T+\tau$ if defined as u on τ. It follows that v, and therefore that $u-v$, has the generalized mean-value property in T. The bounds of $u-v$ in T are then both 0. This proves the theorem.

3.2. The condition as to the existence of v in Theorem IV is satisfied, of course, if we suppose that T is a normal domain, and that u is given as continuous in $T+t$. This suggests a possible result of a much deeper character, namely, that a continuous function having the generalized mean-value property is harmonic if it tends to continuous boundary values at all *regular points** of the boundary of its region of definition. In the next theorem we show that for a bounded function of this type this result is indeed true.

THEOREM V. *Let u be bounded and continuous in $T+\tau$; and let $u(P)=A(u)$ in T. In addition, let $u(P)$ approach at each regular point Q of t a limit value $f(Q)$ when P, while remaining in T, tends to Q. Then, if the values of $f(Q)$ are those of a function continuous on t, it follows that u is harmonic in T.*

The proof rests on the following lemma.

LEMMA. *Let U be continuous in $T+\tau$; and let $U(P)\geqq A(U)$ in T. Let V be harmonic in T. Then $U-V$ tends to its lower bound m in T on a sequence of points in T tending to a point of t.*

The reasoning in this lemma is an extension of that in Theorem I. Let λ be the set of points in T at which $U-V=m$. If λ is void the conclusion follows from the continuity of $U-V$ in T. If λ is not void, and if $c(P)$ is for each point P of λ interior to T, the conclusion follows as in Theorem I. Let us suppose, then, that there is a point P_0 of λ such that $c(P_0)$ has a point Q in common with t. Let W be the function, harmonic in $C(P_0)$, continuous in $C(P_0)+c(P_0)$, which coincides on $c(P_0)$ with U. We observe first that

$$W-V\equiv m \tag{3.21}$$

in $C(P_0)$. In fact, we have

$$W(P_0)-V(P_0)\leqq U(P_0)-V(P_0)=m;$$

so that if (3.21) failed to hold we could select a sequence of points $\{P_n\}$, $n=1, 2, \cdots$, in $C(P_0)$, tending to a point P' of $c(P_0)$, such that

$$\lim_{n\to\infty}\{W(P_n)-V(P_n)\}<m.$$

But

$$\lim_{n\to\infty}U(P_n)=U(P')=\lim_{n\to\infty}W(P_n);$$

* A point Q on the boundary r of a region R, bounded or not, is regular for r if the sequence solution of the Dirichlet problem for R, corresponding to any set of continuous boundary values f, tends to $f(Q)$ at Q. Compare, for example, Wiener, **9**, p. 128, or Kellogg, **4**, p. 606. In **2**, p. 326, Kellogg defines regularity (for three dimensions) by means of the Poincaré-Lebesgue barrier concept. As a consequence of a theorem due to Lebesgue, the complete proof of which can be found in the material in **2**, pp. 326–328, and **4**, pp. 607–609, the barrier definition is equivalent to that given above. The barrier definition is readily extensible to the plane and its equivalence to the sequence solution can be established.

and we should thus have, contrary to the definition of m, $U-V<m$ on some points of $C(P_0)$. The proof is now immediate; for, as a consequence of (3.21), we have

$$\lim_{P\to Q}\{U(P)-V(P)\}=\lim_{P\to Q}\{W(P)-V(P)\}=m\quad (P \text{ in } C(P_0)).$$

3.3. Returning now to the proof of Theorem V, we first observe that it is enough to consider the case in which T lies in the interior S of a circle s of diameter less than 1/2. It is easily seen in fact, on applying a transformation of similitude, that, if the result holds in this case, then it holds in general.*

We let v denote the sequence solution of the Dirichlet problem for T, corresponding to the boundary values f on t; and we let w denote either of the functions, $u-v$ or $v-u$. We observe that, as a consequence of our hypotheses and a familiar property† of the sequence solution, the lower bound m of w in T is finite. To prove the theorem we show that m cannot be negative. We assume the contrary, that m is negative, and arrive eventually at a contradiction.

Let e denote the set of points Q of t at which

$$\varliminf_{P\to Q} w(P)\leqq m/2 \qquad (P \text{ in } T). \tag{3.31}$$

On applying the preceding lemma we see that this set is not void. Plainly, it is bounded and closed. Further, its complement E with respect to the plane is a domain. Since E is open, and contains T, and T is connected, this will follow if we prove that, if Q is any point common to E and the complement of T, then Q can be joined to a point of T by a polygonal line not passing through e. For this, suppose first that Q is a point of $t-e$. In this case the conclusion is immediate; for then Q is at non-zero distance from e and at zero distance from T. Suppose on the other hand that Q is exterior to $T+t$. Let R be a point of t such that no other point of t lies nearer to Q than does R. Then R is regular for t‡; and accordingly,

$$\lim_{P\to R} w(P)=0 \qquad (P \text{ in } T).$$

It follows that R is not a point of e. We conclude that Q can be joined to a

* This reduction is useful only in the plane case.

† The property in question is that the sequence solution lies between the extremes of the assigned boundary values. This results directly from the definition of the sequence solution. See, for example, Wiener, **10**, p. 39, or Kellogg, **2**, pp. 317–326. Kellogg's arguments are for three dimensions. Analogous arguments hold, however, in the plane.

‡ This is a consequence of any one of several criteria for regularity. For references see, for example, Wiener, **9**, pp. 130 and 142.

point of T by a polygonal line of the required type. Accordingly, E is a domain.

We now form the conductor potential ξ of the set e.* We first construct a sequence $\{E_n\}$, $n=1, 2, \cdots$, of normal, unbounded domains, nested in and approximating E.† In particular we construct these domains so that for each n the (finite) boundary e_n of E_n lies in S, and the set E_n+e_n contains no point of e. Then, if T is a three-dimensional domain, we denote by ξ_n the function, harmonic in E_n, which vanishes at infinity and assumes continuously the boundary values 1 on e_n. On the other hand, if T is a plane domain, we first select a point O of e; then denote by $\bar{\xi}_n$ the function, harmonic and bounded in E_n, which assumes continuously the values $\log OP$ on e_n; and finally set

$$\xi_n(P) = \{a_n + \log OP - \bar{\xi}_n(P)\}/a_n,$$

where a_n is the value of $\bar{\xi}_n$ at infinity. In both instances we extend the definition of ξ_n over the points exterior to E_n+e_n by defining it equal to 1 there. Then, at each point P of E the sequence $\{\xi_n(P)\}$ converges. The limit function is the conductor potential ξ.

In regard to ξ we now show that

$$w(P) \geqq m\{1 + \xi(P)\}/2 \tag{3.32}$$

for all P in T. Let n be any positive integer. We observe first that

$$\xi_n \leqq 1 \tag{3.33}$$

in S. This is clear in three dimensions. To see that it is true in the plane we have only to note that

$$a_n < \log 1/2 < 0,$$

as this implies that ξ_n becomes negatively infinite at infinity. We observe next that ξ_n is continuous in S. We see finally that ξ_n is superharmonic in S. In fact, if P is any point of $S\cdot E_n$, or of $S-S\cdot(E_n+e_n)$, then the value of ξ_n at P is equal to the arithmetic mean of its values on every sufficiently small circle about P as center. On the other hand, if P is a point of $S\cdot e_n$, then the value of ξ_n at P exceeds, as we see by applying (3.33), the arithmetic mean of its values on every sufficiently small circle about P as center. We conclude, as a consequence of a familiar theorem‡ on superharmonic functions, that ξ_n is superharmonic in S.

* In this connection see, for example, Wiener, **9**, p. 142, and **10**, p. 26, or for the three-dimensional case, Kellogg, **2**, p. 330. Kellogg's treatment of the conductor potential can be extended to the plane.

† For such a construction see, for example, Kellogg, **2**, pp. 317–323. The author is chiefly interested here in bounded three-dimensional domains. The reasoning, however, is applicable to plane and to unbounded domains.

‡ See, for example, Kellogg, **2**, p. 330.

Consider, then, the function

$$\alpha_n(P) = w(P) - m\{1 + \xi_n(P)\}/2.$$

The function $\pm u - m(1+\xi_n)/2$ satisfies the conditions imposed upon U in our lemma. On the other hand, $\pm v$ satisfies the conditions imposed upon V. Accordingly, α_n tends to its lower bound m_n in T on a sequence of points $\{P_j\}, j=1, 2, \cdots$, in T tending to a point Q of t. Now, if Q is a point of e, we have

$$\varliminf_{j\to\infty} w(P_j) \geqq m, \quad \lim_{j\to\infty} \{-m(1+\xi_n)/2\} = -m.$$

On the other hand, if Q is a point of $t-e$, we have

$$\varliminf_{j\to\infty} w(P_j) \geqq m/2,$$

and also, as we see on applying the fact that $\xi_n \geqq 0$ in S^*,

$$\varliminf_{j\to\infty} \{-m\,(1+\xi_n)/2\} \geqq -m/2.$$

We deduce that $m_n \geqq 0$. Accordingly,

$$w \geqq m(1+\xi_n)/2$$

in T for every n. Allowing n to become infinite, we obtain (3.32).

The proof of the theorem can now easily be completed. We observe first that the capacity of e is 0.† In fact, if its capacity were positive, it would contain at least one point Q regular for the boundary of E.‡ But the point Q, being regular for the boundary of E, would be regular for t since T is contained in E and t contains Q.§ We should therefore have at a point of e

$$\lim_{P\to Q} w(P) = 0 \qquad (P \text{ in } T);$$

and this is impossible. Thus the capacity of e is 0. But now, since the capacity

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* It is plain that this inequality holds if T is a three-dimensional domain. To see that it holds in the plane, one need only apply the formulas given by Wiener in **9**, p. 142. Essentially, it was in order to obtain this inequality that we reduced the problem in the beginning of the proof.

† For the definition of capacity see, for example, Wiener, **9**, p. 143, and **10**, p. 26, or Kellogg, **2**, p. 330.

‡ The lemma that every bounded, closed set of positive capacity contains at least one point regular for the boundary of the unbounded domain bounded by the set is due in the plane to Kellogg, **5**, and in space to Evans, **1**. Evans' proof is valid in the plane.

§ This is immediate in view of the equivalence of the barrier definition to the sequence definition. See, for example, Kellogg, **2**, p. 328. It also follows from Wiener's fundamental criterion on regular points. See Wiener, **9**, pp. 130 and 142.

of e is 0, ξ vanishes identically in E,* and therefore in T. It follows from (3.32) that in T

$$w \geqq m/2.$$

This gives us, as a consequence of the definition of m, the desired contradiction; and this completes the proof.

4.1. **On a construction of the sequence solution of the Dirichlet problem.** The reasoning of Theorem I is applicable in another connection. We close this paper in obtaining by means of it a theorem concerning a method by which the sequence solution of the Dirichlet problem can be constructed. This method was first considered by Lebesgue.† Lebesgue's results were later extended by Perkins.‡

In this theorem we shall suppose that the domains $C(P)$ satisfy the following conditions:

(c) *if U is continuous in $T+t$, then $A\{U(P)\}$ is continuous in T;*

(d) *if U is continuous in $T+t$, then*

$$\lim_{P\to Q} A\{U(P)\} = U(Q) \qquad (P \text{ in } T)$$

at every point Q of t.

We note that these conditions are equivalent to the following:

(e) *if U is continuous in $T+t$, then the function $U_1(P)$, defined as $A\{U(P)\}$ in T, and as $U(P)$ on t, is continuous in $T+t$.*

The theorem is then

THEOREM VI. *Let the domains $C(P)$ satisfy conditions* (c) *and* (d) *above. Let u_0 be continuous in $T+t$; and let*

$$(4.11) \qquad u_n(P) = \begin{cases} A(u_{n-1}), & P \text{ in } T, \\ u_{n-1}(P), & P \text{ on } t, \end{cases} \qquad n = 1, 2, \cdots.$$

Then at every point P of T the sequence $\{u_n(P)\}$ converges to the sequence solution $v(P)$, corresponding to the boundary values u_0 on t, of the Dirichlet problem for T. Further, the convergence is uniform on any closed subset of T.

In regard to the condition (d), we note that (d) is satisfied if the condition (a) of §2.4 holds. In fact, if U is continuous in $T+t$, we can, given P, select points P_1 and P_2 on $c(P)$, such that

$$U(P_1) \leqq A\{U(P)\} \leqq U(P_2).$$

* For the three-dimensional case see, for example, Kellogg, **3**, p. 403. For both cases see Wiener, **9**, p. 142, and **10**, p. 26. Kellogg's proof can be extended to the plane.

† Lebesgue, **7**.

‡ Perkins, **8**.

If, now, (a) holds, and if P tends to a point Q of t, then P_1 and P_2 tend to Q, and $U(P_1)$ and $U(P_2)$ tend to $U(Q)$. Accordingly, $A\{U(P)\}$ tends to $U(Q)$. Thus, (d) can be replaced by (a) in our theorem.

Now, as pointed out before, (a) holds if for each point P of T, $C(P)$ is the interior of a circle about P. Moreover, if in addition we assume that the radius of $c(P)$ is a continuous function of P, then (c) holds. Thus it is enough to assume in the theorem that $C(P)$ is for each P the interior of a circle about P, and that the radius of $c(P)$ is a continuous function of P. A family of circles which satisfies this second condition is that in which the radius of $c(P)$ is, for each P, equal to the distance from P to t. It was with this family of circles that Lebesgue and Perkins were concerned. Lebesgue showed that, if T is a normal domain, the sequence defined in (4.11) converges to the solution, corresponding to the boundary values u_0 on T, of the Dirichlet problem for T. Perkins extended Lebesgue's result to an arbitrary domain, thereby obtaining the result of Theorem VI for Lebesgue's family of circles. In each case the method of proof is somewhat different from ours.

Another point which might be mentioned in connection with the above theorem is that, although we are apparently concerned only with the sequence solution corresponding to values on the boundary which are those of a function continuous throughout $T+t$, this is in reality the general case. Given a function continuous on t, we can, of course, always extend its definition so that the resulting function is continuous on $T+t$.

4.2. We first prove the theorem in the case that u_0 is a superharmonic polynomial. In this case we have $u_0 \geqq A(u_0) = u_1 \geqq m$, where m is the minimum of u_0 in $T+t$, and $u_n \geqq u_{n+1} \geqq m$ if $u_{n-1} \geqq u_n \geqq m$, $n=1, 2, \cdots$. Accordingly

$$u_0(P) \geqq u_1(P) \geqq \cdots \geqq m \tag{4.21}$$

and we can conclude at once that the sequence $\{u_n\}$ converges at each point P of $T+t$ to a limit $u(P)$. We have, then, to show that $u \equiv v$ in T, and that the convergence is uniform in any closed subset of T. Now, of these two propositions, the second follows immediately from the first. In fact, since the u_n are continuous in T and since (4.21) holds, the convergence, by a familiar theorem, is necessarily uniform in any closed subset of T if the limit function is continuous in T. Accordingly, our problem reduces to showing that $u \equiv v$ in T.

Let $T_1, T_2, \cdots$ be a set of normal domains nested in and approximating T. Let v_k be the solution of the Dirichlet problem for T_k, corresponding to the boundary values u_0 on t_k, the boundary of T_k. Let $v_k(P)$ be defined as equal to $u_0(P)$ for P exterior to T_k+t_k. Then v_k is continuous and superharmonic in the plane and we have

$$u_0(P) \geqq v_k(P)$$

in T. Further, at each point P of T, the sequence $\{v_k\}$ converges to $v(P)$. We thus have

$$u_0(P) \geqq v(P) \tag{4.22}$$

in T.

As a consequence of this last inequality and the lemma of §3.2, it is easily seen that

$$u - v \geqq 0 \tag{4.23}$$

in T. In fact, for any fixed integer $n>0$, we have

$$u_n(P) = A(u_{n-1}) \geqq A(u_n)$$

in T. Hence, since u_n is continuous in $T+t$ and v is harmonic in T, the function u_n-v tends to its lower bound in T on a sequence of points $\{P_j\}$, $j=1, 2, \cdots$, in T tending to a point Q of t. Now,

$$\lim_{j\to\infty} u_n(P_j) = u_0(Q), \qquad \overline{\lim_{j\to\infty}}\, v(P_j) \leqq u_0(Q),$$

the latter by (4.22). It follows that

$$u_n - v \geqq 0$$

in T. Allowing n to become infinite, we see that (4.23) holds.

We have now only to prove that

$$v - u \geqq 0 \tag{4.24}$$

in T. For this we consider the function v_k-u corresponding to some integer k. We shall show that v_k-u assumes its lower bound m' in $T+t$ on a point of t. This will of course justify (4.24). In fact, we have

$$v_k - u = u_0 - u_0 = 0$$

on t, and

$$\lim_{k\to\infty} (v_k - u) = v - u$$

in T.

We first observe that, as a consequence of the continuity of the u_n and the fact that (4.21) holds, u is upper semi-continuous in $T+t$. Accordingly, since v_k is continuous in $T+t$, v_k-u is lower semi-continuous there. It follows that the set λ of points in $T+t$, at which $v_k-u=m'$, is not void. To obtain our desired conclusion we prove first that, if P' is a point of $\lambda\cdot T$, then all the points of $c(P')$ are points of λ.

Let U_n, $n=0, 1, \cdots$, be the function, harmonic in $C(P')$, continuous in $C(P')+c(P')$, which assumes the values u_n on $c(P')$. Let V be the function having the corresponding properties with regard to v_k. Then we have

$$U_0 \geqq U_1 \geqq \cdots$$

in $C(P')$, and

$$\lim_{n\to\infty} U_n(P') = u(P') \geqq m \qquad (> -\infty).$$

It follows that the sequence $\{U_n\}$ converges in $C(P')$ to a function U harmonic in $C(P')$. Now, we have

$$V(P') - U(P') \leqq v_k(P') - u(P') = m'.$$

Accordingly, either

$$V - U \equiv m' \tag{4.25}$$

in $C(P')$, or else there is a sequence of points $\{P_j\}$, $j=1, 2, \cdots$, in $C(P')$, tending to a point Q of $c(P')$, such that

$$\lim_{j\to\infty} \{V(P_j) - U(P_j)\} = m' - a,$$

where a is positive. But, if the second of these two alternatives holds, we have

$$\begin{aligned} v_k(Q) - u_n(Q) &= \lim_{j\to\infty} \{V(P_j) - U_n(P_j)\} \\ &\leqq \lim_{j\to\infty} \{V(P_j) - U(P_j)\} \\ &= m' - a \end{aligned} \tag{4.26}$$

for every integer $n \geqq 0$. It follows that (4.25) holds; for otherwise, by (4.26), we should have

$$v_k(Q) - u(Q) < m',$$

contrary to the definition of m'. Let, then, Q be any point of $c(P')$. We have, on applying (4.25),

$$v_k(Q) - u_n(Q) = \lim_{P\to Q} \{V(P) - U_n(P)\} \leqq m' \qquad (P \text{ in } C(P'))$$

for every integer $n \geqq 0$. It follows that $c(P')$ consists wholly of points λ.

We can readily deduce now that there is a point of λ on t. We have only to apply the reasoning of Theorem I. Since $v_k - u$ is lower semi-continuous in $T+t$ and since $T+t$ is closed, the set λ is closed. Hence, since λ is not void, and t is closed, there is a point P of λ whose distance to t is equal to the distance δ from λ to t. If, now, we assume that δ is positive we immediately get a contradiction; for there is a point of $c(P)$ nearer to t than is P and by our previous reasoning all the points of $c(P)$ are points of λ. The theorem for superharmonic polynomials is thus completely established.

4.3. Turning now to the general case, that in which u_0 is given as continuous on $T+t$, we let R denote a closed subset of T and ϵ an arbitrary positive number. To prove the theorem it is enough to show that

$$\overline{\lim_{n\to\infty}} \left| u_n - v \right| \leqq \epsilon$$

uniformly in R.

Now, by the Weierstrass theorem, we can find a polynomial $\bar{u}$ such that

$$\left| u_0 - \bar{u} \right| \leqq \epsilon/2$$

everywhere in $T+t$. Next, we can write

$$\bar{u} = u_0' - u_0''$$

where u_0' and u_0'' are superharmonic polynomials. We set $u_0''' = u_0 - \bar{u}$ and consider the sequences $\{u_n'\}$, $\{u_n''\}$, $\{u_n'''\}$ built upon the continuous functions u_0', u_0'', u_0''' in the same way that $\{u_n\}$ is built upon u_0.

We note first that

$$\left| u_n''' \right| \leqq \epsilon/2$$

in $T+t$ since $|u_0'''| \leqq \epsilon/2$ on t. We note next that, by the conclusion of the preceding paragraph, we have uniformly in R

$$\lim_{n\to\infty} u_n' = v', \qquad \lim_{n\to\infty} u_n'' = v'',$$

where v' and v'' are the sequence solutions of the Dirichlet problem for T corresponding to the boundary values u_0' and u_0''. Thus, since

$$u_n = u_n' - u_n'' + u_n''',$$

we have

$$\overline{\lim_{n\to\infty}} \left| u_n - v' + v'' \right| \leqq \epsilon/2$$

uniformly in R. But, denoting by v''' the sequence solution of the Dirichlet problem for T corresponding to the boundary values u_0''', we have

$$v''' = v - v' + v'', \qquad \left| v''' \right| \leqq \epsilon/2$$

in T. We deduce that uniformly in R

$$\overline{\lim_{n\to\infty}} \left| u_n - v \right| \leqq \overline{\lim_{n\to\infty}} \left| u_n - v' + v''' \right| + \left| v''' \right| \leqq \epsilon .$$

This completes the proof.

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